INSTRUCTOR NOTES
Phase space diagrams: Damped harmonic motion

Emphasis

This tutorial continues the development of phase space diagrams as a tool to represent oscillatory motion. Emphasized in this tutorial are underdamped and critically damped oscillators.

Prerequisites

Students should already have studied simple harmonic oscillators and damped oscillators, and they also need to have completed the tutorial *Phase space diagrams: Simple harmonic motion.* It is also recommended (though not required) that the students have already completed the tutorials *Damped harmonic motion: Motion graphs* and *Damped harmonic motion: Energy loss and the quality factor.*

TUTORIAL PRETEST

In part a of the pretest students are asked to sketch a qualitatively correct phase space diagram for an underdamped oscillator that is released from rest at a position along the $-x$ axis. When administered after completion of the first tutorial on phase space diagrams, most students correctly recognize that the trajectory proceeds in a clockwise direction. Many also indicate correctly on the plot that the amplitude gradually decreases as time goes on. Some students, though, whether in their sketch of the phase space trajectory or in their written answer to the following question, show that they do not recognize that the oscillator must reach a maximum speed before crossing $x = 0$. (This characteristic of the motion is addressed in the tutorial *Damped harmonic motion: Motion graphs,* and for that reason this pretest can actually serve as a <u>post-test</u> for the earlier tutorial on damped oscillators.)

In part b of the pretest students are asked to sketch the phase space trajectory of a critically damped oscillator starting with the same initial conditions as the underdamped oscillator in part a. To answer this question correctly, students must recognize that the asymptotic ($t \to \infty$) behavior of the oscillator can be described by the straight line $\dot{x} = -\gamma x$ in phase space. This line can be drawn accurately because (1) the elliptical trajectory of the corresponding undamped oscillator is shown on the diagram, (2) the "aspect ratio" (b/a) of this trajectory is equal in value to ω_o, and (3) critical damping means $\gamma = \omega_o$. Many students draw incorrect trajectories that show more than one complete cycle of oscillation. Others correctly sketch a trajectory that indicates less than one complete cycle but not the appropriate asymptotic behavior.

TUTORIAL SESSION

Equipment and handouts

Each group will need a whiteboard and set of markers, or a large sheet of paper. Each student will need a copy of the tutorial handout (no special handouts required).

Discussion of tutorial worksheet

Section I: Underdamped oscillator

Students begin by considering the phase space trajectories of two underdamped oscillators, with each having a different quality factor as specified by the factor by which the amplitude decreases with each oscillation. After making a first attempt at each new trajectory (in part A), students check their work in part B by considering whether each trajectory should cross the x- and $\dot{x}$-axes at right angles. By thinking about the direction of the net force on the oscillator when it crosses $x = 0$, students should recognize that the oscillator is always slowing down in such a circumstance; as a result, the phase space trajectory always crosses the $\dot{x}$-axis obliquely. The students should also recognize that each extremum along the x-axis is a turnaround point and thus must correspond to zero velocity, so trajectories always cross the x-axis at right angles. The checkpoint at the end of section I allows instructors to make sure that students understand and can justify these details of each phase space plot.

Section II: Critically damped oscillator

Students continue by considering the motion of a critically damped oscillator. Students differentiate the position function $x(t) = (At + B)e^{-\gamma t}$ of the oscillator and then determine the parametrized equation $\dot{x} = -\gamma x + Ae^{-\gamma t}$. They are guided to recognize that the asymptotic behavior of the oscillator can be described by the straight line $\dot{x} = -\gamma x$ in phase space. Students apply this result by revisiting the second question from the pretest. The checkpoint at the end of the section affords instructors the opportunity to check the reasoning used by the students.

TUTORIAL HOMEWORK

The homework contains two problems that require students to review, apply, and extend the results from this tutorial. Some parts also require the students to synthesize these results with other knowledge about damped oscillators.

1. Students construct an accurate phase space diagram for an underdamped oscillator given its initial conditions and a numerical value for the ratio γ/ω_o. Students must synthesize several pieces of knowledge in order to complete the problem: (i) their knowledge of phase space plots of simple harmonic oscillators, (ii) their knowledge of the relationship between γ, ω_o, and ω_d, and (iii) their knowledge of the amplitude decay of a damped oscillator (*i.e.,* that the value of $\exp[-\gamma T_d]$ is the ratio of successive maxima).

3. Students extend their results by analyzing qualitatively the phase space diagram of an unknown oscillator. They are expected to conclude that the oscillator is overdamped because (a) less than one full cycle of oscillation occurs and (b) the phase space trajectory follows an asymptote whose slope is shallower than (less negative than) the asymptote $\dot{x} = -\gamma x$ for critical damping.

