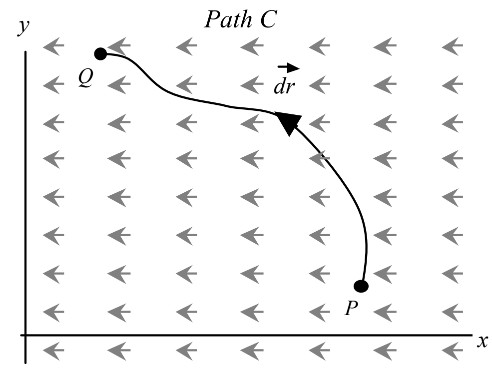
CONSERVATIVE FORCE FIELDS

Conservative forces, such as gravitational forces and elastic (spring) forces, have the special property that the work done by such a force on an object does not depend upon the specific path taken by the object—*i.e.,* the work by conservative forces is said to be *path-independent.* In this tutorial we will develop a mathematical test by which we can tell whether or not a force $\vec{F} = \vec{F}(\vec{r})$ is conservative.

I. Open and closed paths

An object is moved in space from position P to position Q along the path C shown at right. At each location along the path the object experiences a force $\vec{F} = \vec{F}(\vec{r})$ illustrated by the gray vectors drawn at various positions $\vec{r}$.



A. The vector $d\vec{r}$ in the diagram represents an infinitesimal displacement of the object as it is moved along path C.

1. Give a *physical interpretation* to the quantity $\vec{F} \cdot d\vec{r}$ and determine whether that quantity is *positive, negative,* or *zero* evaluated at the location shown.

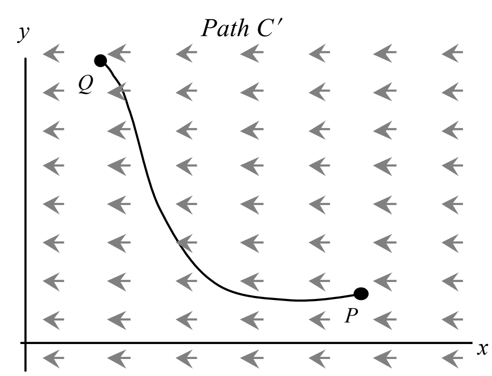
2. Give a *physical interpretation* to the quantity $\int_C \vec{F} \cdot d\vec{r}$.

   As part of your interpretation, sketch additional " $d\vec{r}$ " vectors along path C and explain how those vectors are related to the quantities being added up in the above integral quantity.

   How would (i) the *value* of and (ii) your *interpretation* of the above integral be different if it were evaluated from point Q to point P? Explain. *(Hint:* What would be the significance of reversing the direction of each $d\vec{r}$ vector along path C?*)*

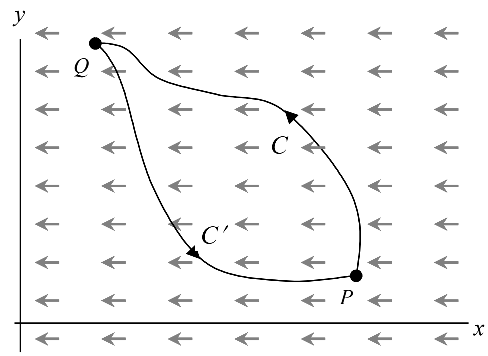
B. Imagine that the object were instead moved from position P to position Q the path C' shown at right.

   If the force $\vec{F}(\vec{r})$ were conservative, how would $\int_{C'} \vec{F} \cdot d\vec{r}$ compare in value to $\int_C \vec{F} \cdot d\vec{r}$? Explain.



C. Imagine now that the object were moved along path C from position P to position Q, and then moved backwards along path C' back to position P.

If the force $\vec{F}(\vec{r})$ were conservative, what can be said about the value of the integral $\oint \vec{F} \cdot d\vec{r}$ evaluated over this entire (closed) path? Explain.



D. If the force $\vec{F}(\vec{r})$ were conservative, would any of your answers in parts B and C change if another pair of positions (other than P and Q) were chosen as the endpoints for paths C and C'? Explain.

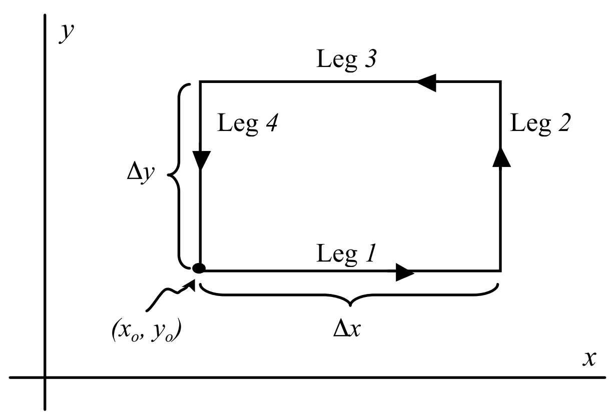
Summarize your results thus far: What can be said about the work done by a conservative force around any closed path? Explain your reasoning.

II. Developing a test for conservative forces

Suppose that you gathered observations by which to express a force $\vec{F}(\vec{r}) = \vec{F}(x, y)$ mathematically as a function of Cartesian coordinates x and y. It is not known whether or not $\vec{F}$ is a conservative force. For the remainder of the tutorial, we develop a test by which to tell whether or not $\vec{F}$ is conservative.

A. We begin by starting small: Consider a small rectangular path with dimensions Δx and Δy. The corner nearest the origin is located at (x_o, y_o). The four segments of the path are labeled "leg *1*" through "leg *4*."

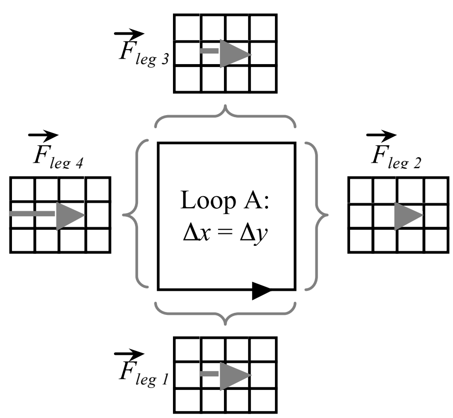
If the force $\vec{F}$ is conservative, what must be true about the work done by that force on an object traversing the entire path?

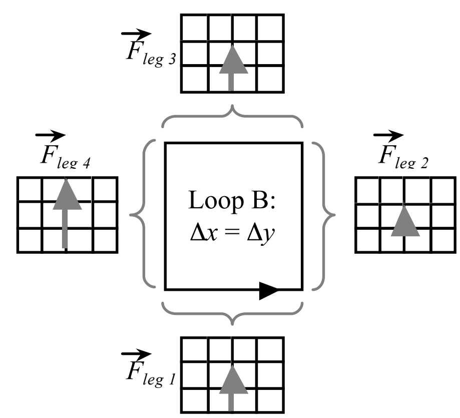


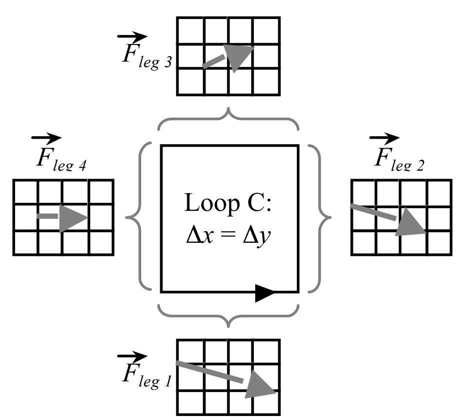
For the rest of this tutorial, we shall restrict our attention to rectangular paths that are so small that the force $\vec{F}$ does not change appreciably over the extent of any of the legs. In other words—taking leg *1* as an example—the value of the force at the midpoint of leg *1* is approximately equal to value of the force at all other points along leg *1*. (Note that this assumption does *not* necessarily mean that $\vec{F}$ has the same value along all four legs.)

B. To practice thinking mathematically about the work done around a loop, consider the three loops below. Each loop is square in shape (*i.e.,* the dimensions Δx and Δy of each are equal) and oriented in the *x*-*y* plane. Beside each leg is drawn a vector representing the force associated with that leg.

   Working under the assumptions outlined above, determine whether the work done along each loop is *positive, negative,* or *zero.* Discuss your reasoning with your partners.

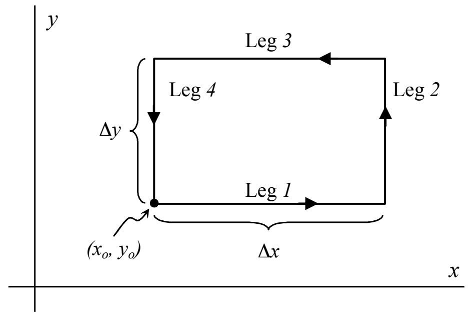






C. Consider again the general case of a *very small* rectangular loop located in the *x*-*y* plane, as shown below right. (*Note:* The loop is very small, so we continue to assume that the value of the force at the midpoint of a leg is approximately equal to value of the force at all other points along that leg.)

1. Which expression below best approximates the work done on an object traversing **leg *1*** of the path? Explain. (*Note:* "$F_x(x, y)$" and "$F_y(x, y)$" denote the functions representing the *x*- and *y*-components of the force.)



a. $F_x\left(x_o + \frac{\Delta x}{2}, y_o\right)\Delta x$

c. $-F_x\left(x_o + \frac{\Delta x}{2}, y_o\right)\Delta x$

b. $F_y\left(x_o + \frac{\Delta x}{2}, y_o\right)\Delta x$

d. $-F_y\left(x_o + \frac{\Delta x}{2}, y_o\right)\Delta x$

2. Write down similar expressions for the work done on the object traversing each of the other legs along the rectangular path. Then, combine your expressions into a single expression that represents the work done on the object traversing the *entire* path.

3. Divide your expression for the work done around the rectangular loop by the total area ($\Delta x \Delta y$). Show that your new expression can be written as a combination of two terms, each with either Δx or Δy in the *denominator* of the term.

For the force $\vec{F}$ to be conservative, what must be true about the value of the expression you have found (work done around the loop divided by the area of the loop)? Explain.

4. Show that, in the limit in which both Δx and Δy tend toward zero, your expression from part 3 can be interpreted as the following combination of partial derivatives evaluated at (x_o, y_o):

$$\frac{Work\,around\,loop}{Area\,subtended\,by\,loop} = \left.\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\right|_{(x,y)=(x_o,y_o)} \qquad \text{Eq. 1}$$

D. Now we are ready to state a general mathematical rule to tell whether or not a force is conservative!

The quantity on the right-hand side of Equation 1, which is expressed in Cartesian coordinates, is known as the (z-component of the) *curl* of the force $\vec{F}$ evaluated at the location (x_o, y_o).

1. On the basis of your results in this tutorial, what must be true about the curl of the force at the location (x_o, y_o) if the force is conservative? Discuss your reasoning with your partners.

2. Although we have not yet proven it (we will in a future homework problem), you probably can guess the correct answer here: Which statement below do you suspect would best describe *any* conservative force $\vec{F}(\vec{r})$? Give as complete an explanation as possible for your answer.

   a. The curl of the force must be non-zero at some locations.

   b. The curl of the force must be non-zero at all locations.

   c. The curl of the force must be zero at some locations.

   d. The curl of the force must be zero at all locations.