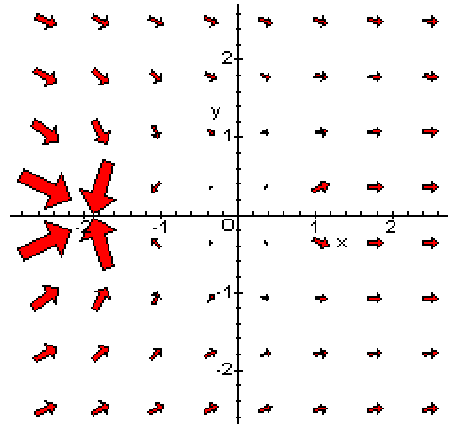
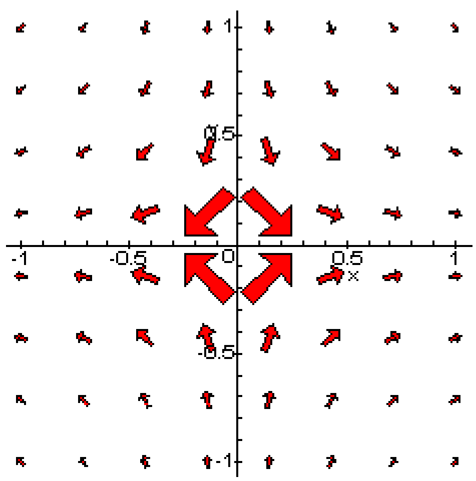
Polya Vector Fields

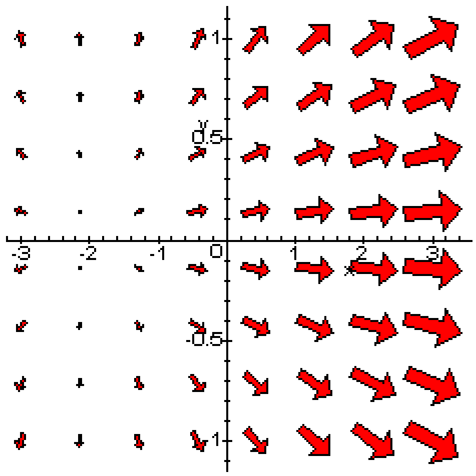


What does the vector field of a complex number represent? What does it tell about the function and the differentiability of the function? Just as an x-y plot of a real function expresses certain characteristics of that function, so too does the graph of a complex function. We will explore vector fields in general and then use Polya's interpretation of vector fields to explore how a vector field tells more about a complex function than a graph tells of a real function.

Let's explore vector fields in general. The vector field for the complex function *f(z)=u+iv* is the set of all vectors <u, v> where u is the real part of the function and v is the imaginary. What does a vector field represent graphically? First, the length of each arrow represents the modulus of the complex number *f(z)*. Thus a zero, a vector where the function *f(z)* = 0, is represented by an arrow of length "zero" (a '+'). The direction of the arrows represents the argument of the function. Consider the following vector fields:



This is the function *f(z)* = 1/z. The arrows represent all the vectors of the function *f(z)* on this interval. This function has a singularity at z=0. The length of each arrow represents the modulus of the function at that point. However, the program we used to draw these vector fields doesn't represent the size of each arrow correctly.



This is a vector field for the function *f(z)* = z + 2. The function equals zero at the point z = -2. If we examine the point z = -2, we see that the vector has length zero and is represented by a '+'.

When dealing with a vector field, there are two terms, curl and divergence that can be used to describe the field. However, before we can successfully define them, we must have a firm understanding of the gradient of a function. The gradient of a function *f(z)* is the vector obtained by evaluating the partial derivatives of *f* at a point. The gradient of a function f(z) = u + *i*v is defined as

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}.$$

With the definition of gradient in mind, we can now proceed with the discussion of curl and divergence. Geometrically, the curl of a vector field tells how the field "swirls" in the vicinity of the point; it tells about the circulation of the field. Curl is calculated by the cross product of the conjugate of the function and the gradient of the function, ∇ x w(z). Divergence of a vector field is a measure of net outflow per unit area. It is a measure of the extent to which a velocity field diverges near the point. Divergence is calculated by the dot product of the conjugate of the function and the gradient of the function, $\nabla \cdot$ w(z). There are two terms used to classify divergence, source and sink. A source, is a point that has what is called "in-degree" of zero. In terms of fluid flow, all flow is out at that point. A sink, on the other hand, is a point that has "out-degree" of zero. In other words, all fluid flow is in at that point.

When is a vector field differentiable? A vector field for the function *f(z)* = u + *i*v is differentiable if the partial derivatives of u and v with respect to x and y exist and are continuous. It is impossible to deduce that a function is differentiable simply by looking at the vector field in general. A differentiable vector field does not imply a differentiable function.

Now let us explore a different aspect of looking at vector fields, Polya's interpretation. Polya's idea was to associate with every function $f(z)=u + iv$, the vector field w(z) = f(z), not *f(z)*. In other words, at each point z in *f*(z), Polya attached the vector with real and imaginary components of the complex conjugate.

Consider what the curl and divergence of the vector field *f(z)* would be. We defined the magnitude of the curl previously as the crossproduct of *w(z)* and the gradient of *w(z)*:

$$
\begin{aligned}
|curl| &= \left|\vec{\nabla} \times \vec{w}\right| \\
&= \left|\frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y}\right| \\
|curl| &= -v_x - u_y
\end{aligned}
$$

When the curl equals zero,

$$u_y = -v_x.$$

From the above calculations we can see that when the curl of a Polya vector field equals zero, one half of the Cauchy Riemann equations is satisfied. We defined the divergence of a vector field as the dot product of *w(z)* and the gradient of *w(z)*:

$$
\begin{aligned}
div &= \vec{\nabla} \bullet \vec{w} \\
div &= \frac{\partial u}{\partial x} + \frac{\partial(-v)}{\partial y} \\
div &= u_x - v_y
\end{aligned}
$$

When the divergence equals zero,

$$u_x = v_y$$

From the calculations above we can see that when the divergence of a Polya vector field equals zero, the other half of the Cauchy Riemann equations is satisfied.

We can conclude that a complex function *f(z)* is differentiable in a region only if its Polya vector field is differentiable, divergence free (incompressible) and curl free (irrotational) throughout the region (Braden 65). By being divergence free and curl free, the divergence and curl of the vector field both equal zero, and thus the Cauchy Riemann equations are satisfied.

It is now possible to deduce something about the differentiability of a function just by looking at its Polya vector field. Graphically when divergence equals zero, you will have no sinks and no sources. Thus if you look at a differentiable Polya vector field that has a sink or a source, it is possible to conclude that the function represented by the vector field is NOT differentiable, because the divergence doesn't equal zero.

This is why Polya chose to use the conjugate of the vector field. The conjugate has many advantages. Consider $f(z) = (u(x,y) + v(x,y))$. Consider an arbitrary point z_o. In order for $f'(z_o)$ to exist, and for the function to be analytic at z_o, the mapping must be differentiable at (x_o, y_o) and the Cauchy Riemann equations must also hold at z_o. What if, instead of *f(z)*, we considered the conjugate *f(z)*. Then we would deal with $f(z) = (u(x,y), -v(x,y))$. Thus *f(z)* would be analytic if the mapping was differentiable and the Cauchy Riemann equations hold true at z_o as well. Here, however, when the curl and divergence of the function are zero, the two Cauchy Riemann equations are satisfied. Thus the conditions for the differentiability of a vector field are lack of divergence, lack of curl, and differentiability.

There is yet one more advantage of using Polya's interpretation of vector fields, and in fact all vector fields in general. We must first discuss singular points of a vector field. Points at which a vector field is zero are called singular points *of the vector field*, because the direction of the vectors changes discontinuously there even though the component functions of w(z) remain differentiable there (Braden 65). Singular points of a vector field can either be a result of a zero of the function f(z), a zero, or a point where f(z) is not differentiable, perhaps a pole.

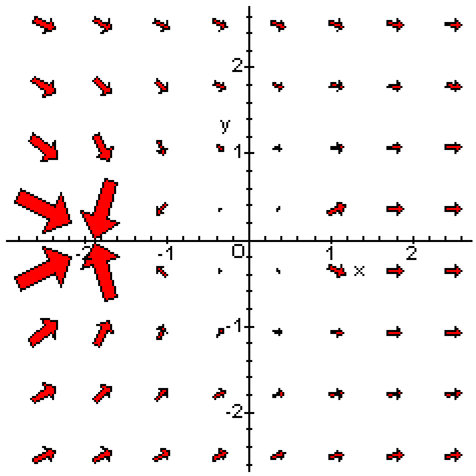
The order of each zero, *k*, can be determined from the vector field, and this concept is one that makes the vector field better than an x-y plot. The vectors w(z) turn through k revolutions clockwise as one traverses a zero of order k in the positive direction. Thus by following a clockwise circular path around a zero and counting how many revolutions each arrow makes, one can decipher the order of the zero. This process is correct due to a result known as the principle of the argument which basically states that the "argument of an analytic function *f(z)* increases by $2p$ times the order of a zero of *f*, as one traverses counterclockwise a small circle around the zero (Banden 65)."

The order of each pole can be determined similarly, however instead of counting the number of clockwise revolutions made by the arrows surrounding the singularity, you count the number of counterclockwise revolutions.

The index of a singularity is just the number of revolutions, clockwise or counterclockwise, that you count when determining the order. Thus if you count three clockwise revolutions of the arrows surrounding a zero of a vector field, you have a zero of order three and the index is three.

Thus we have presented one difference between a real graph and a vector plot, as interpreted by Polya. On a real graph, the order of a zero cannot be determined by looking at the function. The only thing that can be determined is if the zero is odd or even. However, on a vector field, one can determine the actual order of each zero or pole.

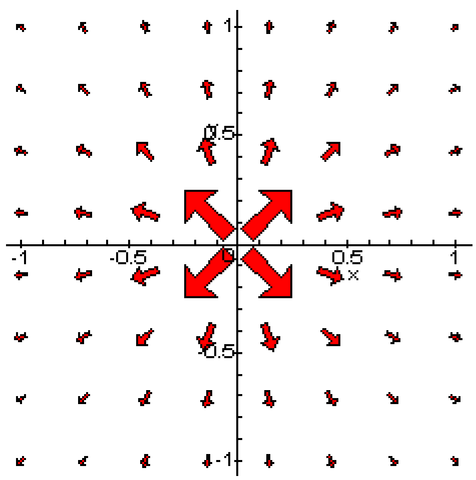
We have looked at vector fields in general, their definition, components and descriptions. We introduced the topics of curl and divergence and their relation to vector fields. We also reviewed the concept of the gradient of the function. Polya introduced a new interpretation of vector fields. Instead of representing the function *f(z)*, Polya used a vector field to describe the conjugate of the function *f(z)*. We used Polya's interpretation of vector fields to deduce the differentiability of a function and the order of each zero of the function. These two aspects cannot be deduced about a real function from its X-Y plot. Thus Polya vector fields have surpassed real graphs in these aspects.



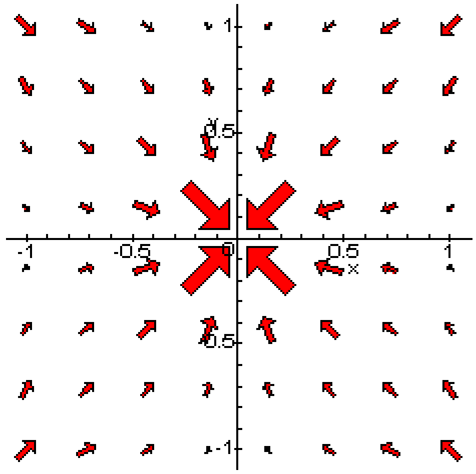
This is a vector plot for the function

$$f(z) = \frac{z^2}{(z+2)(z-4/5)}.$$

Thus the function has singularities at z=-2 and z = 4/5. At z = -2, you can observe how the arrows surrounding that point are changing in direction. By counting the number of revolutions of the arrows, you can figure out the order of the pole, z = -2. The order is 1.



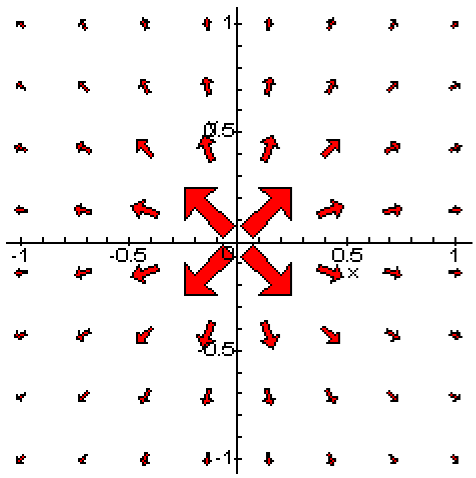
This is a vector plot for the function $f(z) = \frac{1}{z}$. This function is deemed a "source."



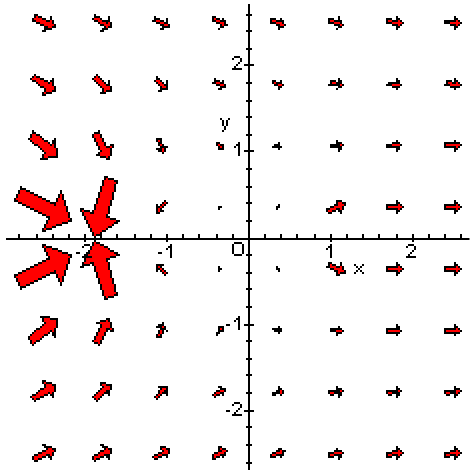
This is a vector plot for the function

$$f(z) = z^3 - \frac{2}{z}.$$

This vector field isn't curl-free. In fact it exhibits a special pattern of curl known as "sink."



This is a vector plot for the function $f(z) = \frac{1}{z}$. This function has a POLE at z=0, because the denominator equals zero.



This is a vector plot for the function $f(z) = \dfrac{z^2}{(z+2)(z-4/5)}$. Thus the function has singularities at z=-2 and z = 4/5. At z = -2, the order equals 1. Thus the INDEX at z=-2 equals 1.

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