Remember in calculus, when integrating a real-valued functions, we were finding the area between the curve and the x-axis. This tool can be used for such things as finding a probability value in statistics, or finding the position of an object given its velocity. Throughout this course, we have learned how to compute complex contour integrals. However, we have not yet learned the meaning behind these complex integrals and their applications. We will explain these mysteries throughout this paper.

Specifically, we would like to focus on the computation of complex integrals in the Polya vector field. To begin we must first understand what a Polya vector field is. Given a function, $f(z) = \mathrm{f}(x+\mathrm{i}y) = u(x,y)+iv(x,y)$, the Polya vector field associated with this function is

$$\overline{f(x+iy)} = < u(x,y), -v(x,y) > .$$

As we can see, this field is simply the field of the conjugate of the function $f(z)$. We will denote the Polya vector field as

$$W(z) = W(x,y) = < w_1(x,y), w_2(x,y) >$$

where $w_1 = u$ and $w_2 = -v$.

Suppose we want to integrate a function f over a simple contour C in the Polya vector field. That is

$$\int_C f(z)dz. \tag{1}$$

Expanding (1) into its real and imaginary parts gives us

$$\int_C (u+iv)(dx+idy). \tag{2}$$

Multiplying the quantities in (2) and separating the result into two integrals, one real and one imaginary, gives us

$$\int_C udx - vdy + i\int_C vdx + udy. \tag{3}$$

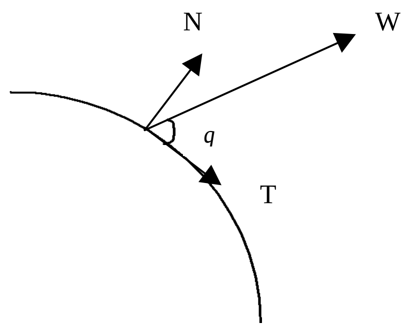
Substituting the components of the Polya vector field into (3) gives us

$$\int_C w_1dx + w_2dy + i\int_C w_1dy - w_2dx. \tag{4}$$

It can be shown that (4) is equal to

$$\int_C W \cdot Tds + i\int_C W \cdot Nds \tag{5}$$

where T is the unit tangent vector, N is the unit normal vector, ds represents the arc length differential, and from above W represents the Polya vector field.



The first integral of (5) represents the flow of the field across the contour C. In other words, this measures how much of the vector field is tangent to the contour C. In a physical sense, this could measure the flow of a fluid along a contour. We know from the formula for the dot product that

$$W \cdot T = |W||T|\cos q = |W|\cos q. \tag{6}$$

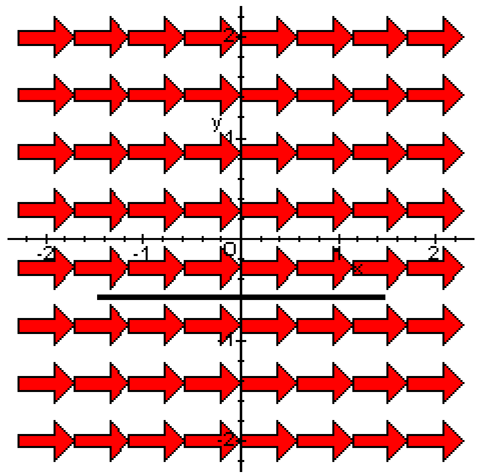
From the graph above, we can clearly see that (6) is just the projection of W onto the unit tangent vector T. (6) computes the flow of the field at a single point on the contour. To calculate the flow across the entire contour, we integrate (6) over the entire length of the contour s.

The second integral of (5) represents the flux of the field across the contour C. That is, the flux measures how much of the vector field is normal to the contour C. Flux can be interpreted physically as the amount of fluid that crosses the contour. From the dot product formula, we know that

$$W \cdot N = |W||N|\cos(90 - q) = |W|\sin q. \tag{7}$$

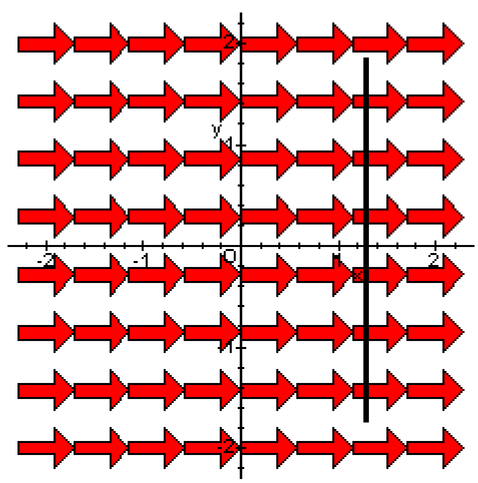
Once again we can see from the graph above that (7) is just the projection of the vector W onto the unit normal vector N. This calculates the flux at a single point on the contour C. To compute the flux across the entire contour we integrate (7) over the length of the contour s.

To illustrate these ideas, we will look at a few examples. First let us consider the case when we have a constant field for the function f(z) = 2 and a line contour which is parallel to the field. This is shown in the graph below.



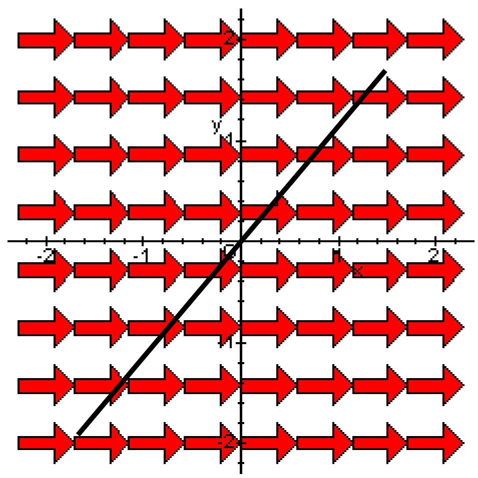
We can see from this graph that the field is tangent to the contour at all points on the contour. So, the dot product of W and T is different from zero. Therefore, the field has flow along the contour. We can also see that the field is perpendicular to the unit normal and therefore the dot product of W and N is equal to zero. Thus we can conclude that the field has no flux across the contour. This makes sense since the vectors do not cross the contour.

Next we will consider the same field which is perpendicular to a vertical line contour. This can be seen in the graph below.



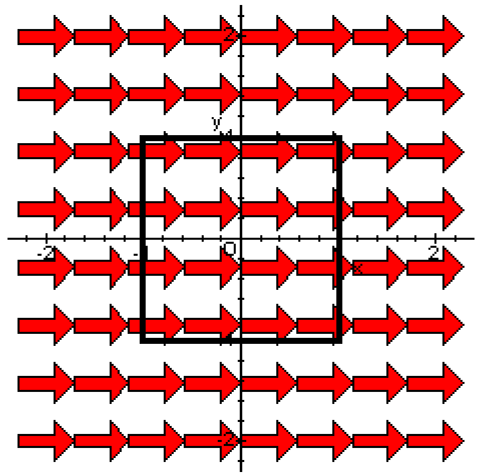
Now the unit tangent of the contour is perpendicular to the field and therefore the dot product of W and T is equal to zero. Thus we know that there is no flow along the contour. However, since the field is parallel to the unit normal, we know that the dot product of W and N is not equal to zero. Hence, the field has flux across the contour.

Again looking at the same field, we will consider an arbitrary line which has a defined slope different from zero. The graph below illustrates this case.



We can deduce from this graph that the field is orthogonal to neither to the unit normal nor the unit tangent of the contour. Therefore, we know that the dot product of W and N and the dot product of W and T are not equal to zero. So, we can conclude that the field has both flux across the contour and has flow along the contour.

Again consider a constant vector field, but now the contour is a square centered at the origin. This closed contour (denoted by *l*) is positively oriented. The field and the contour are shown in the graph below.



To calculate the contour integral along this closed contour, we must find the flux and flow across each line segment and then sum the results.

First consider the vertical line on the right which will be denoted as l_1. The field is orthogonal to the unit tangent of l_1 and parallel to the unit outward normal. So, the field has flux across this line segment, but no flow along this line segment. The field is flowing in the same direction as the unit normal, so the angle between W and N is zero. Recall that W=2, therefore, we know that (5) is

$$i\int_{l_1} 2 \cdot 1\cos(0)ds$$

$$= 2i^*s, \qquad (8)$$

where s is the length of the line segment.

Now consider the vertical line on the left which will be denoted as l_3. Once again we can see that the field is orthogonal to the unit tangent of l_3 and parallel to the unit outward normal, but it is flowing in the opposite direction of the unit normal. Therefore, the angle between W and N is equal to π. So, the field has flux across this line segment, but no flow along this line segment. Now we know that (5) equals

$$i\int_{l_3} 2 \cdot 1\cos(p\ )ds$$

$$= -2i^*s. \qquad (9)$$

Next, looking at the top line segment which we will call l_2, we can see that the field is parallel to the unit tangent (but in the opposite direction) and perpendicular to the unit outward normal. So, we know that the field has flow along the line segment, but no flux across the line segment. Therefore, we know that (5) equals

$$\int_{l_2} 2 \cdot 1 \cos(\pi) ds$$

$$= -2*s. \qquad (10)$$

Finally, consider the bottom line segment which we will call l_4. Once again we can see that the field is parallel to the unit tangent and perpendicular to the unit outward normal. So, the field has flow along the line segment, but no flux across the line segment. Thus, we know that (5) is equal to

$$\int_{l_4} 2 \cdot 1 \cos(0) ds$$

$$= 2*s. \qquad (11)$$

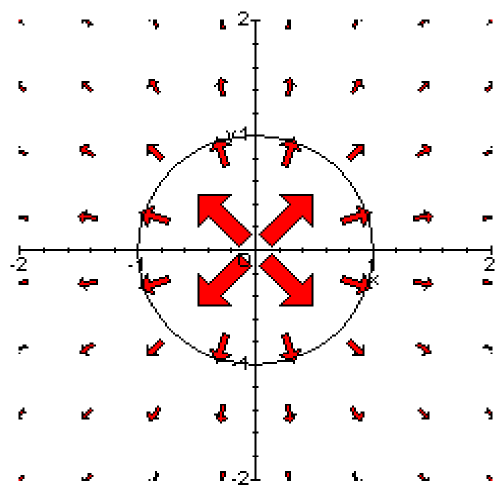
Summing (8), (9), (10), and (11) we get

$$2i*s + (-2i*s) + (-2*s) + 2*s = 0.$$

This is equal to the complex integral along the contour l. Since the field is analytic everywhere inside l, this result is consistent with the Cauchy-Goursat Theorem. This theorem states that if f is analytic in a simply connected domain D, and C is a simple closed contour that lies in D, then

$$\int_C f(z) dz = 0.$$

Next, we will look at one final example. We want to consider the case when there is a singularity inside a closed contour. Therefore, let us look at the example of when the Polya vector field is for f(z) = 1/z and the contour is the unit circle centered at the origin. This case is illustrated in the graph below.



The field is normal to the contour at every point, thus there is flux across the contour but no flow along the contour. It can be shown that the magnitude of W on the contour is equal to 1. Note that the graph above is not drawn to scale. We know that (5) equals

$$i\int_C 1 \cdot 1\cos(0)ds$$

$$= i*\text{s}$$

$$= 2\pi i.$$

This result is consistent with the Integral of Basic Powers Theorem. This theorem states that

$$\int_{|z-a|=R} (z-a)^n\, dz = \begin{cases} 2pi \text{ if } \; n = -1 \\ 0 \text{ otherwise} \end{cases}.$$

In conclusion, we have shown that the value of the complex contour integral represents (flow of the field along the contour) + i*(flux of the field across the contour). Through, examples we have seen that if the value of the complex integral is real, then the field has only flow along the contour. However, if the value of the complex integral is imaginary, then the field has only flux across the contour. If the value of the complex integral is complex, then the field has both flux across the contour and flow along the contour.

Interpretation of the Complex Contour Integral

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